Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Pure and Applied Algebra 200 (2005) 137 – 147
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Every group is an outer automorphism group of a finitely generated group ☆

Inna Bumagin[a, b, *], Daniel T. Wise[a]

[a] *Department of Mathematics, McGill University, Montreal, Que., Canada H3A 2K6*
[b] *School of Mathematics and Statistics, Carleton University, Ottawa, Ont., Canada K1S 5B6*

Received 3 October 2002; received in revised form 20 December 2004
Communicated by J. Rhodes
Available online 25 February 2005

Abstract

We show that every countable group Q is isomorphic to $\mathrm{Out}(N)$ where N is a finitely generated subgroup of a countable $C'(\frac{1}{6})$ small-cancellation group G. Furthermore, when Q is finitely presented, we can choose G to be finitely presented and residually finite.

MSC: 20F28; 20F06; 20F67; 20E26; 20E22

1. Introduction

While many groups arise as automorphisms of algebraic objects, it is not true that every group Q arises as $\mathrm{Aut}(N)$ for some group N. Indeed, it is well-known that a nontrivial cyclic group of odd order is not isomorphic to $\mathrm{Aut}(N)$ for any group N. The situation for $\mathrm{Out}(N)$ is markedly different and the question of whether a given group Q can be realized as an outer automorphism group of some group N has attracted some recent attention.

In [11] it was shown that any group Q is isomorphic to $\mathrm{Out}(N)$ where N is the fundamental group of a graph of groups but the cardinality of the generating set of N is greater than the

☆ Research supported by NSERC grant.
* Corresponding author. School of Mathematics and Statistics, Carleton University, Ottawa, Ont., Canada K1S 5B6. Fax: +1 613 520 3536.
E-mail addresses: bumagin@math.carleton.ca (I. Bumagin), wise@math.mcgill.ca (D.T. Wise).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2004.12.033

cardinality of the generating set of Q. The problem becomes more complicated if N is required to belong to a special class of groups. Any Q is isomorphic to $\mathrm{Out}(N)$ where N is simple [2], or where N is torsion-free metabelian with trivial center [4]. Any countable Q is isomorphic to $\mathrm{Out}(N)$ where N is a locally finite p-group [3]. Finally, when Q is finite, N can be chosen to be the fundamental group of a closed hyperbolic 3-manifold [8].

The first main result in this paper is Theorem 11, where we show that any countable Q is isomorphic to $\mathrm{Out}(N)$, where N is a finitely generated subgroup of a finitely generated $C'(\frac{1}{6})$ group G. The second main result is Theorem 15 where we show that if Q is finitely presented, N can be chosen to be a finitely generated subgroup of a finitely presented residually finite $C'(\frac{1}{6})$ group. We note that in analogy with Kojima's result, when Q is finite, the group N produced by our construction is the fundamental group of a finite $C'(\frac{1}{6})$ complex.

The main idea is to use a variant of Rips's construction [12] in order to obtain a short exact sequence $1 \to N \to G \to Q \to 1$, where G is a $C'(\frac{1}{6})$ group and N is finitely generated. As observed in [17], the natural homomorphism $Q \to \mathrm{Out}(N)$ is easily shown to be injective in this case. The main difficulty is to choose G so that there are no "surprise" outer automorphisms of N, and so $Q \to \mathrm{Out}(N)$ is surjective. A criterion for establishing this surjectivity is given in Lemma 9, which is the central underlying technical result in this paper.

We close this introduction by discussing the extent to which our results can be generalized. Since there are uncountably many finitely generated groups Q, it is not true that every such Q arises as $\mathrm{Out}(N)$ where N is a subgroup of a finitely presented group. Nevertheless we pose the following problem that would generalize our two main results:

Problem 1. Is every countable group Q isomorphic to $\mathrm{Out}(N)$ where N is finitely generated and residually finite?

2. Review of small-cancellation theory

A *piece* in a presentation $\langle a_1, a_2, \ldots \mid R_1, R_2, \ldots \rangle$ is a word U such that $US_1 \neq US_2$ and both $(US_1)^{\pm 1}$ and $(US_2)^{\pm 1}$ are cyclic permutations of relators.

The presentation satisfies $C'(\alpha)$ if for any piece U which is a subword of the cyclic word UR_j, we have $|U| < \alpha |R_j|$ where we use $|X|$ to denote the length of the word X. We will also say that the set of words $\{R_1, R_2, \ldots\}$ in the free group $\langle a_1, a_2, \ldots \mid - \rangle$ satisfies $C'(\alpha)$ if the presentation above satisfies $C'(\alpha)$. We say that G is a $C'(\alpha)$ group if G has a presentation that satisfies the $C'(\alpha)$ condition.

We will need the following well-known facts about the *metric small cancellation groups* whose presentations satisfy a $C'(\alpha)$ condition. The first theorem is the key result in small cancellation theory.

Proposition 2 (*Lyndon and Schupp [10, Chapter V, Theorem 4.4]*). *If W is a freely reduced word which represents the identity element in a $C'(\frac{1}{6})$ group, then W contains a subword V such that VS is a conjugate of some relator or its inverse, and S is the concatenation of at most three pieces.*

A proof of the following theorem for $C'(\frac{1}{8})$ groups, based on an idea of Lipschutz [9] was given by Lyndon and Schupp [10, Chapter V, Theorem 10.1]. For $C'(\frac{1}{6})$ groups it was proved by Greendlinger [5].

Proposition 3. *Every torsion element in a $C'(\frac{1}{6})$ small-cancellation group is a power of a conjugate of an element W, where W^n is a relator.*

Greendlinger proved for $C'(\frac{1}{8})$ groups [6], and later for $C'(\frac{1}{6})$ groups [7], that in these groups, any two elements that commute necessarily belong to a cyclic subgroup. Truffault [15] and Seymour [13] independently, strengthened the result of Greendlinger and showed that in $C'(\frac{1}{6})$ groups, centralizers of non-trivial elements are cyclic. From this latter result we observe that

Corollary 4. *Let G be a $C'(\frac{1}{6})$ group, and let N be a non-cyclic subgroup of G. Then the centralizer of N in G is trivial.*

Proof. Let $N = \langle h_1, h_2, \ldots\rangle$. Then $C_G(N) \subset C_G(h_1) \cap C_G(h_2) \cap \cdots$ Let $d \in C_G(h_1) \cap C_G(h_2) \cap \cdots$ hence $h_i \in C_G(d)$ for all i. If $d \neq 1$, then $C_G(d)$ is cyclic so that H is cyclic, a contradiction. Thus, $C_G(N) = \{1\}$. □

2.1. Construction of special small-cancellation words

Definition 5. An x, y word is *special* if it is a positive word in x, y with no x^2 or y^3 subwords.

It is easy to provide sets of special words satisfying stringent small cancellation conditions. The following lemma provides us with examples of such sets; we will use these examples in our proof.

Lemma 6. *Consider the infinite word $W = xy(xy^2)xy(xy^2)^2xy(xy^2)^3\cdots$*

(1) *for each infinite sequence $\{n_1, n_2, \ldots\}$ of natural numbers, one can find an infinite family $\mathscr{W}_\infty = \{w_1, w_2, \ldots\}$ of words which satisfies the $C'(\frac{1}{20})$ condition, so that $|w_i| \geqslant n_i$ for each i.*
(2) *for each positive integer m and a large enough integer p one can find a family $\mathscr{V}_m = \{v_1, v_2, \ldots, v_m\}$ of words which satisfies the $C'(\frac{1}{20})$ condition, so that given a subset $\mathscr{I}$ of $\{1, 2, \ldots, m\}$, the lengths of the elements of $\mathscr{V}_m$ are as follows: $|v_i| = p$ for each $i \in \mathscr{I}$ and $|v_i| = p + 1$ for each $i \notin \mathscr{I}$.*

Proof. To prove the first assertion, let

$$w_i = xy(xy^2)^{k_i}xy(xy^2)^{k_i+1}\cdots xy(xy^2)^{2k_i}, \tag{1}$$

where $k_1 \geqslant \max\{60, \sqrt{\frac{2}{3}n_1}\}$ and $k_i \geqslant \max\{2k_{i-1}, \sqrt{\frac{2}{3}n_i}\}$ be the words in $\mathscr{W}_\infty$. Obviously, w_i is a special x, y word of length $|w_i| = \frac{9}{2}k_i^2 + 5k_i + 1 > n_i$, and the length of the maximal piece $p_i = y(xy^2)^{2k_i-1}xy(xy^2)^{2k_i}xy$ of w_i equals $|p_i| = 12k_i + 2$. Hence $\mathscr{W}_\infty$ satisfies the $C'(\frac{1}{20})$ condition, and we have proven the first assertion.

To prove the second assertion, consider the finite subset

$$\mathscr{W}_m = \{w_2, w_3, \ldots, w_{m+1}\} \subset \mathscr{W}_\infty$$

chosen according to part (1) of this lemma (we set $n_i = 1$ for all i). We modify the words $w_i \in \mathscr{W}_m$ as follows. Let $p > |w_{m+1}|$ be an integer number. A "correcting" exponent for $i = 2, 3, \ldots, m+1$, is the maximal integer c_i which does not exceed $(p - |w_i|)/(2(k_i + 1))$. Define

$$v_i = (xy)^{c_i}(xy^2)^{k_i}(xy)^{c_i}(xy^2)^{k_i+1}\cdots(xy)^{c_i}(xy^2)^{2k_i}x^{\varepsilon_1^{(i)}}(xy)^{\varepsilon_2^{(i)}}(xy^2)^{\varepsilon_3^{(i)}},$$

where $\varepsilon_1^{(i)}, \varepsilon_3^{(i)} \in \{0, 1\}$ and $\varepsilon_2^{(i)}$ is a nonnegative integer, so that $|v_i| = p$. Note that the length of the correcting term $x^{\varepsilon_1^{(i)}}(xy)^{\varepsilon_2^{(i)}}(xy^2)^{\varepsilon_3^{(i)}}$ does not exceed $2(k_i + 1)$.

We claim that $\mathscr{V}_m = \{v_1, v_2, \ldots, v_m\}$ satisfies the $C'(\frac{1}{20})$ condition. Indeed, the values of c_i are all different, hence the maximal length of a piece p_i that is a subword of v_i is bounded by $|p_i| \leqslant |(xy^2)^{2k_i-1}(xy)^{c_i}(xy^2)^{2k_i}| < 2c_i + 12k_i$. Since $|v_i| > \frac{9}{2}k_i^2 + 2c_ik_i + 2c_i$, we have

$$\frac{|v_i|}{|p_i|} > \frac{\frac{9}{2}k_i^2 + 2c_ik_i + 2c_i}{2c_i + 12k_i} = \frac{\frac{9}{4}(k_i/c_i)k_i + k_i + 1}{1 + 6k_i/c_i}.$$

It can be readily seen that $|p_i|/|v_i| < \frac{1}{20}$ for all $i > 1$. The claim follows. □

3. Outer automorphism groups and short exact sequences

The plan is to use the natural homomorphism $Q \to \mathrm{Out}(N)$, which exists for any short exact sequence

$$1 \to N \to G \to Q \to 1. \tag{2}$$

Specifically, G acts by conjugation on N, so there is a homomorphism $\hat{\phi} : G \to \mathrm{Aut}(N)$, and obviously $\hat{\phi}(N) \subset \mathrm{Inn}(N)$, so there is an induced homomorphism $\phi : Q \to \mathrm{Aut}(N)/\mathrm{Inn}(N) = \mathrm{Out}(N)$.

Our goal is as follows: for a given group Q, find G and N in (2) so that

$$\phi : Q \to \mathrm{Out}(N) \tag{3}$$

is an isomorphism.

3.1. The map ϕ is injective

We will prove that ϕ is injective by applying the following simple criterion which holds for an arbitrary extension (2).

Lemma 7. *If the centralizer $C_G(N)$ of N in G is contained in N, then ϕ is injective.*

Proof. The kernel of ϕ is represented by elements $g \in G$ which act on N like an inner automorphism, say conjugation by $n \in N$. Specifically, for each $x \in N$, we have $x^n = x^g$ and so $x^{ng^{-1}} = x$. Thus ng^{-1} is in the centralizer of N, and so $ng^{-1} \in N$, and hence $g \in N$ as claimed. Thus g represents a trivial element of Q. $\square$

In our construction, G will be a non-cyclic small cancellation group and N will be a 2-generated subgroup of G. Therefore, by Corollary 4, the centralizer of N is trivial, so it is obviously contained in N, and we have the following statement.

Lemma 8. *If the group G in* (2) *is non-cyclic and $C'(\frac{1}{6})$, then ϕ is injective.*

3.2. The map ϕ is surjective

Given a group Q presented by $\langle \mathscr{A} \mid \mathscr{R} \rangle$, where $\mathscr{A} = \{a_i\}_{i \in \mathscr{I}}$ and $\mathscr{R} = \{R_j\}_{j \in \mathscr{J}}$, we form a presentation

$$\begin{aligned}&\langle x, y, \mathscr{A} \mid x^{a_i} = A_i,\ x^{a_i^{-1}} = B_i,\ y^{a_i} = C_i,\ y^{a_i^{-1}} = D_i,\ i \in \mathscr{I},\ R_j = E_j,\\ &\quad j \in \mathscr{J},\ x^p,\ y^p,\ xyxy^2xy^3 \cdots xy^q \rangle,\end{aligned} \tag{4}$$

where the A_i, B_i, C_i, D_i and E_j are words in x and y that are specified in Lemma 9(2) below. Letting G denote the group presented above, we obtain a short exact sequence $1 \to N \to G \to Q \to 1$ where $N = \langle x, y \rangle$. The first four types of relations guarantee that N is normal in G, and the $R_j = E_j$ relations guarantee that $G/N \cong Q$. The last three relations are added to facilitate the arguments determining $\mathrm{Out}(N)$ that we shall make below. Of course, a priori, there is no reason to conclude that N is nontrivial.

In order to obtain a group G with the desired properties, we use small cancellation techniques.

Lemma 9. *If Presentation* (4) *for group G in* (2) *satisfies the following conditions, then the map $\phi : Q \to \mathrm{Out}(N)$ is surjective.*

(1) *Presentation* (4) *satisfies the $C'(\frac{1}{11})$ condition.*
(2) *The relations of G are chosen so that A_i, B_i, C_i, D_i and E_i are special x, y words, and each such special word has length $\geqslant \frac{30}{31}$ of its corresponding relator.*
(3) *Exponent q is an integer satisfying $q > 45$.*
(4) *Exponent p is a number satisfying $p > 2q$.*

Proof. Pick $\alpha \in \mathrm{Aut}(N)$. Since the only relators in G which are proper powers are x^p and y^p, by Proposition 3, the only torsion elements in G are conjugates of powers of x or y. Therefore, $\alpha(x) = g_1^{-1} z_1^{p_1} g_1$, and $\alpha(y) = g_2^{-1} z_2^{p_2} g_2$, where each $g_i \in G$, each $z_i \in \{x, y\}$, and $0 \leqslant p_i < p$ for each $i = 1, 2$.

The subgroup N has many more relations than just x^p and y^p and in particular, is not a free product. For instance $W = xyxy^2xy^3 \cdots xy^q$ is a relation which does not hold between torsion elements in a free product of cyclic groups. We shall first prove that $g_1 = g_2$, in other words we show that the image of α in $\mathrm{Out}(N)$ is conjugation by g_1 (or g_2) pre-composed by a map θ that sends x to some power of x (or y), and sends y to some power of y (or x).

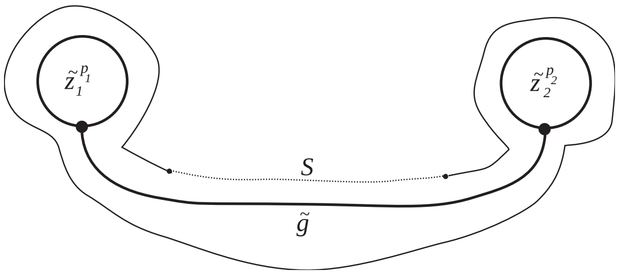

Fig. 1. $\alpha(N)$ is the fundamental group of the bold graph. The thin path represents a relator *VS*.

We shall later prove that $\theta \notin \mathrm{Aut}(N)$ unless $\theta(x) = x$ and $\theta(y) = y$. Our arguments show that in all the other cases the subgroup $\langle \alpha(x), \alpha(y) \rangle$ is isomorphic with $\langle \alpha(x) \rangle * \langle \alpha(y) \rangle$ and therefore cannot be isomorphic to N.

We argue as follows. We may assume that the words $g_1^{-1} z_1^{p_1} g_1$ and $g_2^{-1} z_2^{p_2} g_2$ are freely reduced. Suppose that $g_1 \neq g_2$. If $g_1 g_2^{-1} = z_1^{q_1} z_2^{q_2}$ for some integers q_1, q_2, then $g_1 = z_1^{q_1} z_2^{q_2} g_2$, hence $g_1^{-1} z_1^{p_1} g_1 = g_2^{-1} z_2^{-q_2} z_1^{p_1} z_2^{q_2} g_2$. We see that α is conjugation by g_2 precomposed by conjugation by $z_2^{q_2}$ and by the map θ; in this case, we can proceed to the next step. Therefore, we may assume that $g_1 g_2^{-1} \neq z_1^{q_1} z_2^{q_2}$ for any integers q_1, q_2. Consider the free group $F = \langle \tilde{x}, \tilde{y}, \tilde{a}_1, \tilde{a}_2, \dots \mid - \rangle$ and the homomorphism $\eta : F \to G$ that sends $\tilde{x}$ to x, $\tilde{y}$ to y and $\tilde{a}_i$ to a_i for each $i \in \mathscr{I}$. Pick the shortest $\tilde{g}, \tilde{z}_1, \tilde{z}_2 \in F$ so that $\eta(\tilde{g}) = g_1 g_2^{-1}$, $\eta(\tilde{z}_i) = z_i$, and consider the subgroup $\tilde{K}$ of F generated by $\tilde{z}_1$ and $\tilde{g} \tilde{z}_2 \tilde{g}^{-1}$. We claim that the image $K = \eta(\tilde{K})$ in G is a free product of two cyclic subgroups and therefore cannot be isomorphic to N. Indeed, consider the graph Γ (see [14]) illustrated in Fig. 1 corresponding to the generators of $\tilde{K}$. The graph Γ is represented by the bold part of Fig. 1. It has two closed paths (labelled by $\tilde{z}_1^{p_1}$ and $\tilde{z}_2^{p_2}$), which are connected by an arc labelled by $\tilde{g}$. In the product $\tilde{g} \tilde{z}_2 \tilde{g}^{-1}$, absorb the last $\tilde{z}_2$ syllable of $\tilde{g}$ if necessary, and change the base point of Γ if $\tilde{g}$ starts with $\tilde{z}_1$. By the above observations, we can assume that $\tilde{g}$ starts with a letter different from $\tilde{z}_1$ and ends by a letter different from $\tilde{z}_2$. If the intersection $\tilde{K} \cap \ker(\eta)$ is not trivial, then by Proposition 2, there is $\tilde{k} \in \tilde{K}$ that contains a subword V such that *VS* is a conjugate of some relator or its inverse, and S is the concatenation of at most three pieces. In particular, this large part V of a relator is represented by a path in the graph. Unless V is either x^n or y^n for some $0 < n < p$, the word V contains a long sequence of alternating powers of x and y. Therefore, the path V necessarily travels through the entire arc $\tilde{g}$. We distinguish the following two cases.

(1) V traverses $\tilde{g}$ at most once (Fig. 1).
(2) V traverses $\tilde{g}$ more than once.

In the first case, $\tilde{g}$ is not a geodesic. Indeed, add an arc labelled by S to Γ so as to obtain the relator $U = VS$ as the label of a closed path in the resulting graph. Since V traverses $\tilde{g}$ at most once, the complement of $\tilde{g}$ in R consists of at most z_1^2, z_2^2, S and two pieces of U. Therefore,

$$|\tilde{g}| \geqslant |U| - |S| - 4 - \tfrac{2}{11}|U| > |U| - 4 - \tfrac{5}{11}|U| > \tfrac{5}{11}|U|;$$

the last inequality holds for $U > 44$. Hence, $\tilde{g}$ can be replaced by the shorter path which consists of S and two pieces.

In the second case, observe that our choice of the relations implies that V cannot travel through the entire arc $\tilde{g}$ more than twice. Indeed, each occurrence of $\tilde{g}$ in V is followed by an occurrence of $\tilde{g}^{-1}$. Since both x and y appear in the relators only with positive exponents, and $\tilde{g}$ is nontrivial, $\tilde{g}$ necessarily contains $\tilde{a}_i^{\pm 1}$ for some i. The structure of relators implies that

$$V = \tilde{g}_1 z_{i_1}^{\pm 2} \tilde{g}^{\pm 1} z_{i_2}^{\pm 2} \tilde{g}^{\mp 1} z_{i_1}^{\pm 2} \tilde{g}_2,$$

where $\tilde{g}_1$ and $\tilde{g}_2$ are subpaths of $\tilde{g}$. Observe that $\tilde{g}$, $\tilde{g}_1$ and $\tilde{g}_2$ are pieces. Altogether we have $|V| < \frac{4}{11}|U| + 6$ and since $6 < \frac{1}{11}|U|$, we obtain the following inequality $|V| < \frac{5}{11}|U|$ that contradicts Proposition 2. Therefore, each $\tilde{k} \in \tilde{K} \cap \ker(\eta)$ contains more than half of x^r or y^r. Let $\tilde{w} \in F$ be a word of length at least two in the free product $\langle \tilde{z}_1 \rangle * \langle \tilde{g}\tilde{z}_2\tilde{g}^{-1} \rangle$, and let $\tilde{w} = \tilde{w}_1 \cdots \tilde{w}_s$ be its normal form. Our arguments imply that $\eta(\tilde{w})$ is non-trivial in G unless $\eta(\tilde{w}_i) = 1_G$ for each $i = 1, \ldots, s$. The claim follows.

We have shown that $\alpha(x) = g_1^{-1} z_1^{p_1} g_1$, and $\alpha(y) = g_1^{-1} z_2^{p_2} g_1$. We now show that $p_1 = p_2 = 1$. We are choosing shortest representatives subject to $0 < p_i < p$. Notice that $z_1 \neq z_2$, for otherwise $\alpha(N)$ is cyclic and since by Lemma 10 below, N is not cyclic, α is not an automorphism. First, suppose that $z_1 = x$, so that $z_2 = y$. In other words, we suppose that α sends x to a power of x and y to a power of y (up to conjugacy). The image of each relator of G under α is necessarily trivial. Let W denote the last relator in Presentation 4. Consider

$$W' = g_1 \alpha(W) g_1^{-1} = x^{p_1} y^{p_2} x^{p_1} y^{2p_2} \cdots x^{p_1} y^{qp_2}.$$

Since $\alpha(W) =_G 1$, we have $W' =_G 1$, and Proposition 2 implies that W' contains more than half of a relator. Since we have chosen shortest representatives for $\alpha(x)$ and $\alpha(y)$, the word W' does not contain more than half of the relator x^p. The other relators are words with no x^n subwords for $|n| > 1$; therefore, $p_1 = 1$. Note that if we started with a negative value of p_1, then we would actually have $p + p_1 = 1$, so that $p_1 = -(p-1)$. But then $g_1^{-1} x g_1$ should be the shortest representative for $\alpha(x)$. Furthermore, in all cyclic conjugates of $A_i^{\pm 1}$, $B_i^{\pm 1}$, $C_i^{\pm 1}$, $D_i^{\pm 1}$ and $E_i^{\pm 1}$ the generator y appears with exponents ± 1 and ± 2. Assume that the exponents of x and y in W' are positive. For each m with $1 < m < q$, we have that $(m-1)p_2, mp_2, (m+1)p_2 \equiv_p 1$ or 2. So either $p_2 \equiv_p 0$ (which is impossible) or $2p_2 \equiv_p 0$. In this latter case, we also have that $p_2 \equiv_p \pm 1$, therefore $p = 2$, but our construction implies that $p \gg 2$. Hence, we have to use W when shortening W'. In W, each two sequential powers of y differ by 1, and their values increase, hence $p_2 = 1$.

Now, assume $z_1 = y$ so that $\alpha(x) = g_1^{-1} y^{p_1} g_1$, and $\alpha(y) = g_1^{-1} x^{p_2} g_1$. Then

$$W' = g_1 \alpha(W) g_1^{-1} = y^{p_1} x^{p_2} y^{p_1} x^{2p_2} \cdots y^{p_1} x^{qp_2} = 1.$$

Apply x^p whenever it is possible. As we showed above, we still remain with different exponents of x, for instance each three sequential exponents of x in W' are different. In the rest of the relations x appears with the exponent 1 only, therefore large subwords of these relations cannot occur in W'. Hence by Proposition 2, $W' \neq 1$, therefore, $\alpha(W) \neq 1$, a contradiction.

In conclusion, we have shown that $\alpha(x) = g_1^{-1}xg_1$ and $\alpha(y) = g_1^{-1}yg_1$, so α is induced by conjugation by an element of G as claimed. $\square$

Lemma 10. *If a group G has a presentation satisfying the conditions of Lemma* 9, *then the subgroup $N = \langle x, y\rangle$ is not cyclic so that G is not cyclic.*

Proof. The freely reduced word $xyx^{-1}y^{-1}$ is too short to contain a large part of any relation. Therefore by Proposition 2, we conclude that x and y do not commute, and so N cannot be cyclic, hence G is not cyclic. $\square$

3.3. Countable groups are outer automorphism groups of finitely generated groups

We apply Lemma 9 to prove the following theorem.

Theorem 11. *Every countable group Q is isomorphic with* $\mathrm{Out}(N)$ *where N is a finitely generated subgroup of a countable $C'(\frac{1}{6})$ group.*

Proof. Let $Q = \langle a_1, a_2, \ldots \mid R_1, R_2, \ldots\rangle$. We consider the group G presented by (4). We choose relations in Presentation (4) so that G satisfies the $C'(\frac{1}{11})$ condition. Also, we have the short exact sequence (2), where $N = \langle x, y\rangle$. Our goal is to show that the map $\phi : Q \to \mathrm{Out}(N)$ is an isomorphism.

In order to define the relations of G, we define an auxiliary sequence of integers $\{n_j\}$ by $n_j = 30|R_i|$ for $j = 5i$ and $n_j = 1$ for $j \neq 5i$, $i = 1, 2, \ldots, \infty$ and use this sequence to choose a sequence of words $\mathscr{W}_\infty = \{w_1, w_2, \ldots\}$ according to Lemma 6; the family $\mathscr{W}_\infty$ satisfies the $C'(\frac{1}{20})$ condition. Let

$$\mathscr{V} = \{A_1, B_1, C_1, D_1, E_1, A_2, B_2, C_2, D_2, E_2, \ldots\}$$

be an ordered set of symbols. We assign values to the symbols from $\mathscr{V}$ according to the natural one-to-one correspondence between $\mathscr{V}$ and $\mathscr{W}_\infty$, so that $A_1 = w_1$, $B_1 = w_2$, $C_1 = w_3$ etc. In particular, our choice of the values of E_i implies that $|E_i| \geqslant 30|R_i|$, hence the length of a maximal piece of the relation $R_i^{-1}E_i$ is bounded above by $(\frac{1}{30} + \frac{1}{20})|w_{5i}| > |R_i| + |p_i|$. We conclude that the family $\{R_i^{-1}E_i \mid i = 1, 2, \ldots, \infty\}$ satisfies the $C'(\frac{1}{11})$ condition. Let $q = 45$, and let p be a number bigger than twice the maximal exponent of y that occurs in the relations A_i, B_i, C_i, D_i, E_i and $xyxy^2xy^3 \cdots xy^{45}$; for instance $p = 97$. Clearly, the presentation of G satisfies the $C'(\frac{1}{11})$ condition, in particular, G is a $C'(\frac{1}{6})$ group. Furthermore, by Lemma 10, G is not cyclic. Therefore, ϕ is injective by Lemma 8. It is readily seen that the presentation of G satisfies the conditions of Lemma 9, hence ϕ is surjective. $\square$

4. Finitely presented groups are outer automorphism groups of finitely generated residually finite groups

We begin this section with the following lemma, which will be used to prove the residual finiteness of our groups.

Lemma 12. *Let G be presented by $\langle x, y, a_1, \dots a_r \mid V_1, \dots, V_n\rangle$, where $V_1, \dots, V_n$ are reduced and cyclically reduced words. Let a homomorphism $\psi : \langle x, y, a_1, \dots, a_k\rangle \to \mathbb{Z}$ be such that $\psi(x) = \psi(y) = 1$, and $\psi(a_i) = 0$ for each i. Suppose that x and y only appear positively in the relators V_j. Suppose that $\psi(V_j)$ is constant and $\psi(V_j) > \frac{8}{9}|V_j|$ and suppose that the presentation for G satisfies the $C'(\frac{1}{9})$ condition. Then G is residually finite.*

Lemma 12 will be deduced from a result of [16] which we shall state after the following definition.

Definition 13 (*Wise [16]*). Fix a homomorphism $\psi: \langle a_1, \dots, a_r \mid -\rangle \to \mathbb{Z}$. Let $\|\psi\| = \max\{|\psi(a_i)| : 1 \leqslant i \leqslant r\}$. Let V be a word in $\{a_1^{\pm 1}, \dots, a_r^{\pm 1}\}$. Let subwords$(V)$ denote the set of subwords of the cyclic word V, so each element of subwords(V) is a subword of some cyclic permutation of V. We define peak(V) to be $\max\{-\psi(U) : U \in \text{subwords}(V)\}$. Note that peak$(V) \geqslant 0$ because subwords(V) contains the empty word. The number peak(V) bounds the extent to which V can backtrack relative to ψ. For example, if ψ is the exponent sum homomorphism then $\|\psi\| = 1$, and if $V = a^{-1}b^3a^{-2}b^3aba^{-3}$ then $\psi(V) = 2$ and peak$(V) = 4$. Finally, we define $\delta(V)$ to be 0 or 1 depending upon whether $\psi(V)$ is even or odd.

We quote the following from [16, Theorem 3.6]:

Theorem 14. *The group $\langle a_1, \dots, a_r \mid V_1, \dots, V_s\rangle$ is residually finite provided that the following conditions hold*:

(1) *Each V_i is reduced and cyclically reduced and the presentation satisfies $C'(1/n)$.*
(2) *There is a homomorphism $\psi: \langle a_1, \dots, a_r \mid\rangle \to \mathbb{Z}$ such that $\psi(V_1) = \psi(V_2) = \cdots = \psi(V_s) \geqslant 1$.*
(3) *There exists ρ such that $\|\psi\| \leqslant \rho + 1$, and $\rho \geqslant \text{peak}(V_i)$ and $\psi(V_i) > 4\rho + 1$ for each i.*
(4) *For each i there are constants $\lambda_i \geqslant 1$ and ε_i such that $\lambda_i|U| + \varepsilon_i \geqslant |\psi(U)|$ for each $U \in \text{subwords}(V_i)$.*
(5) *For each i we let $\delta(V_i)$ be 0 or 1 in case $\psi(V_i)$ is even or odd. Finally, the following equation is satisfied for each i*:

$$\frac{\psi(V_i) - \delta(V_i)}{2} - 2\rho - \varepsilon_i \geqslant \frac{4\lambda_i}{n}|V_i|.$$

We note that hidden in the conditions of the above theorem are two implicit inequalities. Firstly, $\varepsilon_i \geqslant 0$ since we can apply condition (4) to the empty word. Secondly, $n \geqslant 8$ is less obvious but holds by applying conditions (3) and (4) to Eq. (5).

Now we can prove Lemma 12.

Proof. We deduce the statement of Lemma 12 from Theorem 14. Conditions (1) and (2) of Theorem 14 are a part of our assumptions. Note that peak$(V_j) = 0$ and $\|\psi\| = 1$, hence if we set $\rho = 0$, then (3) will hold. $\psi(V_j)$ is equal to the sum of the lengths of the subwords of V_j formed by x and y, because x and y only appear positively in the relators V_j and the exponents of a_i do not affect the value of $\psi(V_j)$. Hence, for $\lambda_i = 1$ and $\varepsilon_i = 0$ inequality

(4) holds. Plug the values of ρ, λ_i, ε_i and $n = 9$ into Eq. (5) and obtain

$$\frac{\psi(V_i) - \delta(V_i)}{2} \geqslant \frac{4}{9}|V_i|,$$

which follows immediately from our assumptions. Hence by Theorem 14, G is residually finite. □

The following theorem is the main result of this section.

Theorem 15. *Every finitely presented group Q is isomorphic to* $\mathrm{Out}(N)$ *where N is a finitely generated subgroup of a finitely presented residually finite $C'(\frac{1}{11})$ group.*

Proof. Given a group Q presented by $\langle a_1, a_2, \ldots, a_k \mid R_1, R_2, \ldots, R_t\rangle$, we form a presentation of G as follows (cf. (4)):

$$\begin{aligned}&\langle x, y, a_1, a_2, \ldots, a_k \mid x^{a_i} = A_i,\ x^{a_i^{-1}} = B_i,\ y^{a_i} = C_i,\ y^{a_i^{-1}} = D_i,\\ &\quad R_i = E_i,\ x^p,\ y^p,\ W\rangle,\end{aligned} \tag{5}$$

where A_i, B_i, C_i, D_i, E_i and W are positive words in x, y, and p is a sufficiently large number (which we specify below). Thus we obtain a short exact sequence $1 \to N \to G \to Q \to 1$ (cf. (2)) where $N = \langle x, y\rangle$. Define a homomorphism $\psi\colon G \to \mathbb{Z}$ by $\psi(x) = \psi(y) = 1$ and $\psi(a_j) = 0$ for each j (cf. Lemma 12). Therefore, for each $g \in G$ which we consider a word in the alphabet of the generators of G, the value $\psi(g)$ is the sum of the exponents of x and y which occur in g.

To specify the relators of G, let $\mathscr{W}_\infty = \{w_1, w_2, \ldots\}$ be the family of x, y words chosen according to part (1) of Lemma 6, and let $m = 4k + t$. Let a positive integer p satisfy the following inequalities: $p > |w_{m+1}|$ and $p > 30\max_{1 \leqslant i \leqslant t}|R_i|$; also we assume that $p = q(q+3)/2$ for some integer q. Let $\mathscr{I} = \{1, 2, \ldots, t\}$, and let $\mathscr{V}_m$ be the family of words obtained from the finite subset $\mathscr{W}_m$ of $\mathscr{W}_\infty$, according to part (2) of Lemma 6. Fix a one-to-one correspondence between the finite set of variables $\{A_j,\ B_j,\ C_j,\ D_j,\ E_i\}$ and the finite set of words $\mathscr{V}_m(x, y) = \{v_1, \ldots, v_m\}$ and assign values to $A_j,\ B_j,\ C_j,\ D_j$ and E_j according to this correspondence. The only requirement is that the following condition is satisfied: $|A_j| = |B_j| = |C_j| = |D_j| = p$ and $|E_j| = p + 1$. Let

$$W = xyxy^2xy^3 \cdots xy^q,$$

where q is the integer chosen above (it is easy to see that $q \gg 45$). Obviously, this presentation of G satisfies the conditions of Lemma 12, hence G is residually finite. Moreover, the presentation of G satisfies the conditions of Lemma 9, hence ϕ is surjective. By Lemma 10, G is not cyclic, hence by Lemma 8, ϕ is injective. Therefore, Q is isomorphic to $\mathrm{Out}(N)$ where N is the subgroup of G generated by x and y. □

In conclusion, we note that if N is finitely generated and residually finite then $\mathrm{Aut}(N)$ is residually finite [1].

References

[1] G. Baumslag, Automorphism groups of residually finite groups, J. London Math. Soc. 38 (1963) 117–118.

[2] M. Droste, M. Giraudet, R. Göbel, All groups are outer automorphism groups of simple groups, J. London Math. Soc. (2) 64 (3) (2001) 565–575.

[3] M. Dugas, R. Göbel, On locally finite *p*-groups and a problem of Philip Hall's, J. Algebra 159 (1) (1993) 115 –138.

[4] R. Göbel, A.T. Paras, Outer automorphism groups of metabelian groups, J. Pure Appl. Algebra 149 (3) (2000) 251–266.

[5] M. Greendlinger, On Dehn's algorithms for the conjugacy and word problems, with applications, Comm. Pure Appl. Math. 13 (1960) 641–677.

[6] M. Greendlinger, A class of groups all of whose elements have trivial centralizers, Math. Z. 78 (1962) 91–96.

[7] M. Grindlinger, The problem of conjugacy and coincidence with an anticenter in the theory of groups, Sibirsk. Mat. Ž. 7 (1966) 785–803.

[8] S. Kojima, Isometry transformations of hyperbolic 3-manifolds, Topology Appl. 29 (3) (1988) 297–307.

[9] S. Lipschutz, On powers of elements in *S*-groups, Proc. Amer. Math. Soc. 13 (1962) 181–186.

[10] R.C. Lyndon, P.E. Schupp, Combinatorial Group Theory, Springer, Berlin, 1977 Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[11] T. Matumoto, Any group is represented by an outer automorphism group, Hiroshima Math. J. 19 (1) (1989) 209–219.

[12] E. Rips, Subgroups of small cancellation groups, Bull. London Math. Soc. 14 (1) (1982) 45–47.

[13] J.E. Seymour, Conjugate powers and unique roots in certain small cancellation groups, Ph.D. Thesis, University of Illinois, 1974.

[14] J.R. Stallings, Topology of finite graphs, Invent. Math. 71 (3) (1983) 551–565.

[15] B. Truffault, Centralisateurs des éléments dans les groupes de Greendlinger, C.R. Acad. Sci. Paris. Sér. A 279 (1974) 317–319.

[16] D.T. Wise, Residual finiteness of quasi-positive one-relator groups, J. London Math. Soc. (2) 66 (2) (2002) 334–350.

[17] D.T. Wise, A residually finite version of Rips's construction, Bull. London Math. Soc. 35 (1) (2003) 23–29.